Relativistic analysis of an earth-satellite time transfer

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Abstract

Analytical treatment of time transfer problem for Earth-Satellite system is presented. The development was made in a complete relativistic framework. In accordance with modern clock precision and for low altitude orbits, we neglect the other bodies and consider only the $\frac{1}{c^2}$ Earth potential developed up to the J_2 term in spherical harmonics.

1 Introduction

During the last few years, new technological developments increase significantly the time transfer accuracy using laser links. Further technological developments will go beyond the precision of the currently used relativistic theoretical description generally used in data reduction programs. At present time, only the first order in the post newtonian metric is considered and the Earth potential is reduced to the monopolar term, which corresponds to a spherical Earth (Blanchet & al. 2001).

In foreseeable future laser links between ground based stations and low orbit satellites (T2L2, ACES) would reach precisions of order 10^{-14} or even 10^{-15} s. In this context, we show modelizing relativistic effects considering spherical Earth only is not sufficient. Hence, we consider the other effects to select the ones which have to be taken into account at this level of accuracy.

In sec. 2, we derive the orders of magnitude of the different relativistic effects, related to both Earth's potential and external bodies. We show, only the Earth's J_2 potential term has to be included. In sec. 3, we give the analytical expressions of the trajectory and the corresponding propagation time.

2 Selecting the relevant terms for low orbit satellites

In general relativity, photons follow null geodesics of the space-time metric $g_{\mu\nu}$, which encodes the gravitational interaction. Following the IAU2000 convention (Soffel & al. 2003), we take the following form for the metric (one uses units such that $G = c = 1$)

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$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \equiv -(1 - 2U)dt^2 + (1 + 2U)\delta_{ij}dx^i dx^j. \qquad (1)$$

This metric is developed up to first order in the potential U, which represents the sum of the newtonian form of the potentials associated to each gravitational source. This form of the metric will turn out to be sufficient for the proposed application. The sources of gravitation are the Earth and external bodies. As usual, greek indices $(\alpha, \beta,...)$ run from 0 to 3 (space-time variables), while latin indices $(i, j, k,...)$ run from 1 to 3 (spatial variables).

The Earth potential can be developed in spherical harmonics. The monopolar term (spherical Earth) writes

$$U_{E,m} = \frac{M_E}{r}$$

where M_E is Earth's mass. Beyond the monopolar term, the dominant term is the so called J_2-term, which, in brief, corresponds to the Earth's oblateness contribution to the gravitational field. This term is of the order of

$$U_{E,J_2} \sim J_2 \frac{M_E R_E^2}{r^3}$$

where R_E is Earth's radius. The other terms in the multipolar development are at least two orders of magnitude weaker. The (tidal) contribution of each external body (essentially the Moon and the Sun) to the potential is of the order of

$$U_{ext} \sim x^i x^j \partial_i \partial_j U \sim \left(\frac{r}{L}\right)^2 \frac{M}{L}$$

where M and L are the mass and the distance of the involved body.

$U_{E,m}$ has relative effects of the order of 10^{-9} on the motion of a photon. Hence, for a $\sim 1000\ km$ altitude satellite, the effect on the flying time of a photon linking a ground based station and this satellite can reach $\sim 10^{-11}$ or some $10^{-12}\ s$. Since J_2 is of order 10^{-3}, the contribution of this term is of order 10^{-14} or some $10^{-15}\ s$. The contribution of the other multipolar terms is then at best of order $10^{-16}\ s$. It turns out that the contributions of external bodies are of the order of the J_2 contribution for altitudes corresponding to geostationary satellites. Besides, for $r < 17000\ km$, it would be meaningless taking external perturbations into account if harmonic terms beyond J_2 are not included in the Earth's potential model. Second order terms in the metric (neglected in the metric presented) are of order 10^{-18}, hence induce time delay corrections of order $10^{-20}\ s$.

The precision which could be reached by time transfer experiments in a foreseeable future is of the order of $10^{-15}\ s$. Hence, considering satellites at altitudes $\sim 1000\ km$, it is necessary to include, beyond the Earth's monopolar term, the Earth's J_2 contribution to the potential, but it is legitimate to neglect all the other contributions.

3   Obtaining the time transfer

As usual in relativistic solar system experiments, we take the following form of the metric (constants G and c have been explicitly included):

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \equiv -\left(1 - \frac{2U}{c^2}\right)c^2 dt^2 + \left(1 + \gamma\frac{2U}{c^2}\right)\mid d\vec{r}\mid^2 \qquad (2)$$

$$U = \frac{GM_E}{r} + J_2\frac{GM_E}{2r}\left(\frac{R_E}{r}\right)^2\left(1 - 3\left(\frac{z^2}{r^2}\right)\right)$$

which generalizes the (1) metric, in order to include viable alternative geometric gravity theories, like scalar-tensor theories (Will 1993). The deviation of those theories to general relativity is encoded in the γ term which is unity in general relativity. Current experimental tests and astronomical observations show $\mid \gamma - 1 \mid$ is less than some 10^{-5} (Will 2006). Hence, a possible deviation from general relativity should not have measurable effects in our problem. Since making computations in this more general context does not lead to dramatic complications, we present results in the general case, but it is always possible to set $\gamma = 1$ if the general relativity case is considered. The light geodesic equation writes :

$$\frac{\mathrm{d}k_\alpha}{\mathrm{d}\lambda} = \frac{1}{2}k^\mu k^\nu \partial_\alpha g_{\mu\nu} \qquad (3)$$

with $k_\alpha = g_{\alpha\beta}k^\beta$, $k^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}$, λ being an affine parameter along the geodesic. The wave vector k^α is isotropic which means $k^\alpha k_\alpha = 0$. At first order in the potential, the geodesic equation leads to:

$$\frac{\mathrm{d}\overset{(\rightarrow 1)}{k_0}}{\mathrm{d}\lambda} = 0 \qquad (4)$$

$$\frac{\mathrm{d}\overset{(\rightarrow 1)}{k_i}}{\mathrm{d}\lambda} = (1 + \gamma)\frac{GM_E}{c^2}\left[\partial_i\frac{1}{r} + \frac{J_2}{2}R_E{}^2\partial_i\frac{1}{r^3} - \frac{3}{2}J_2 R_E{}^2\partial_i\frac{z^2}{r^5}\right] \qquad (5)$$

where $\overset{(\rightarrow 1)}{Q^\alpha} \equiv \overset{(0)}{Q^\alpha} + \overset{(1/2)}{Q^\alpha} + \overset{(1)}{Q^\alpha}$ is the development of Q^α up to the first order in U of any quantity Q^α. Up to first order in U the solution reads :

$$\overset{(\rightarrow 1)}{x^0} = x_0^0 + f(\vec{x}, \vec{n})\lambda + c^{-1}g(\vec{x}, \vec{n}) \qquad (6)$$

$$f(\vec{x}, \vec{n}) = \left(1 + 2\frac{GM_E}{c^2}J_2 R_E^2\left\{A\frac{1}{r} + B\frac{1}{r^3}\right\}\right)$$
$$g(\vec{x}, \vec{n}) =$$
$$2\frac{GM_E}{c^2}\left[ln\left(\frac{r + \vec{n}\cdot\vec{x}}{r_0 + \vec{n}\cdot\vec{x_0}}\right) + J_2 R_E^2\left\{A \ \vec{n}\cdot\vec{x_0}\left(\frac{1}{r} - \frac{1}{r_0}\right) + B \ \vec{n}\cdot\vec{x_0}\left(\frac{1}{r^3} - \frac{1}{r_0^3}\right) + C \ \left(\frac{1}{r^3} - \frac{1}{r_0^3}\right)\right\}\right]$$

A, B, C are constant along the geodesic, x_0^α correspond to initial conditions and $\vec{n}$ is a normalized tri-vector ($\vec{n}\cdot\vec{n} = 1$) related to the components of the wave vector k^α. We get from this formulae the function $\overset{(\rightarrow 1)}{\lambda}(x^0, \vec{n})$, from which we obtain $\overset{(\rightarrow 1)}{x^i}\left(\overset{(\rightarrow 1)}{\lambda}(x^0, \vec{n})\right)$ and find (writing, as usual, $x^0 = t$):

$$\overset{(\rightarrow 1)}{x^i}(t, \vec{n}) = x_0^i + n^i c(t - t_0) + \frac{GM_E}{c^2}f^i(t - t_0, \vec{n}) \quad (7)$$

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$$f^i(t - t_0, \vec{n}) = (1 + \gamma) \left[-n^i ln \left(\frac{r + \vec{x} \cdot \vec{n}}{r_0 + \vec{x_0} \cdot \vec{n}} \right) - B^i(r - r_0) + J_2 R_E^2 \ \ h^i(t - t_0, \vec{n}) \right]$$

$$h^i(t - t_0, \vec{n}) = \Omega^i \left(\frac{t - t_0}{r^3} - \vec{n} \cdot \vec{x_0} \left(\frac{1}{r^3} - \frac{1}{r_0^3} \right) \right) + \Delta^i \left(\frac{t - t_0}{r} - \vec{n} \cdot \vec{x_0} \left(\frac{1}{r} - \frac{1}{r_0} \right) \right)$$

$$+ \Xi^i \left(\frac{1}{r^3} - \frac{1}{r_0^3} \right) + \Theta^i \left(\frac{1}{r} - \frac{1}{r_0} \right) + \Gamma^i(r - r_0)$$

with $B^i, \Omega^i, \Delta^i, \Xi^i, \Theta^i, \Gamma^i$ constant along the geodesic. Obtaining the time transfer is then solving the equation $\overset{(\rightarrow 1)}{x^i}(t, \vec{n}) = \overset{(\rightarrow 1)}{x^i_{sat}}(t)$, where $\overset{(\rightarrow 1)}{x^i_{sat}}(t) \equiv x^i_{s0} + v^i_s(t_0)(t - t_0) + \frac{a^i_s}{2}(t_0)(t - t_0)^2$. We get $\overset{(\rightarrow 1)}{T} \equiv \overset{(0)}{T} + \overset{(1/2)}{T} + \overset{(1)}{T}$ with $T = t - t_0$:

$$\overset{(0)}{T} = c^{-1} \sqrt{(\vec{x_{s0}} - \vec{x_0})^2} \quad (8)$$

$$\overset{(1/2)}{T} = \left(\overset{(0)}{\vec{n}} \cdot \frac{\vec{v_s}}{c} \right) \overset{(0)}{T} \quad (9)$$

$$\overset{(1)}{T} = \frac{1}{2} \left[\left(\frac{\vec{v_s}}{c} \right)^2 + \left(\overset{(0)}{\vec{n}} \cdot \frac{\vec{v_s}}{c} \right)^2 + \left(\overset{(0)}{\vec{n}} \cdot \frac{\vec{a_s}}{c} \right) \overset{(0)}{T} \right] \overset{(0)}{T} - \frac{GM_E}{c^3} \left(\overset{(0)}{\vec{n}} \cdot \vec{f} \left(\overset{(0)}{T}, \overset{(0)}{\vec{n}} \right) \right) \quad (10)$$

with $\overset{(0)}{n^i} = \frac{x^i_{s0} - x^i_0}{\sqrt{(\vec{x_{s0}} - \vec{x_0})^2}}$.

4   Conclusion

As a consequence of the loss of spherical symmetry, a lot of new terms appears from the J_2 potential term (see 7). Possible next steps in order to upgrade our analysis could involve the next order in the metric development ($g_{0i} = O(3/2)$) and/or the next terms in the spherical harmonic development ($J_3, C_{22}, ...$). Thus, to go further, the next development could be based on a $3/2$ order metric with the earth potential developed up to the J_2 term or more if needed. External potentials are not relevant for low altitude orbits and should be neglected in future studies.

References

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